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1999 J. Phys. A: Math. Gen. 32 5715

(http://iopscience.iop.org/0305-4470/32/30/314)

J. Phys. A: Math. Gen. **32** (1999) 5715–5734. Printed in the UK PII: S0305-4470(99)02978-9

Strong coupling perturbation expansions for anharmonic oscillators. Numerical results

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Received 25 March 1999

Abstract. The strong coupling expansion coefficients for the ordinary and renormalized energies of the ground and first excited states of the quartic, sextic, octic and decadic anharmonic oscillators with the Hamiltonian $H = p^2 + x^2 + \beta x^{2m}$, $m = 2, 3, 4, 5$ are computed. The expansion coefficients are also computed for higher excited states of the quartic oscillator. The large-order behaviour of the coefficients, the radii of convergence of the series and the summation rules for the coefficients are discussed. It is shown that, in contrast to the divergent weak coupling expansions, the renormalized strong coupling perturbation wavefunctions have simple form and straightforward physical interpretation. Finally, both the strong coupling perturbation approaches are compared.

1. Introduction

In this paper, we investigate the Schrödinger equation

$$H\psi = E(\beta)\psi \tag{1}$$

for the anharmonic oscillators, where

$$H = p^2 + x^2 + \beta x^{2m} \qquad \beta \geqslant 0 \quad m \geqslant 2 \tag{2}$$

and $p = -\mathrm{i}\mathrm{d}/\mathrm{d}x$.

As is well known, the ordinary energy $E(\beta)$ can be expressed as a weak coupling perturbation series in powers of β,

$$E(\beta) = \sum_{n=0}^{\infty} b_n \beta^n \tag{3}$$

which diverges for every $\beta > 0$ (see, e.g., [1–7]). The energy $E(\beta)$ also possesses the strong coupling expansion (see, e.g., [2, 3, 6])

$$E(\beta) = \beta^{1/(m+1)} \sum_{n=0}^{\infty} K_n \beta^{-2n/(m+1)} \tag{4}$$

corresponding to an equivalent Hamiltonian

$$H = \beta^{1/(m+1)}(p^2 + \beta^{-2/(m+1)}x^2 + x^{2m}) \tag{5}$$

which can be obtained from equation (2) by means of the scaling transformation $x \to \beta^{-1/[2(m+1)]}x$. The series (4) converges if β is sufficiently large i.e. if $\beta > \beta_{min}$, where $\beta_{min} > 0$.

0305-4470/99/305715+20$30.00

Alternative perturbative approaches based upon renormalization (Wick ordering [8] or scaling [5, 9–12]) have considerable advantages. The renormalization can be introduced by means of a new coupling constant κ related to β by the equation [9–12]

$$\beta = \frac{\kappa}{B_m(1-\kappa)^{(m+1)/2}} \tag{6}$$

where

$$B_m = m(2m-1)!!/2^{m-1}. \tag{7}$$

This transformation maps the original unbounded interval $\beta \in [0, \infty)$ onto the bounded interval $\kappa \in [0, 1)$ (see also [13–16]). With the help of equation (6), the Hamiltonian (2) can be expressed in terms of a renormalized Hamiltonian H_R

$$H = \frac{H_R}{(1-\kappa)^{1/2}} \tag{8}$$

where

$$H_R = p^2 + x^2 + \kappa(x^{2m}/B_m - x^2) = p^2 + x^{2m}/B_m + (1-\kappa)(x^2 - x^{2m}/B_m). \tag{9}$$

The renormalized energy

$$E_R(\kappa) = (1-\kappa)^{1/2} E(\beta) \tag{10}$$

can either be expressed as a renormalized weak coupling expansion in κ

$$E_R(\kappa) = \sum_{n=0}^{\infty} c_n \kappa^n \tag{11}$$

or as a renormalized strong coupling expansion in $1-\kappa$

$$E_R(\kappa) = \sum_{n=0}^{\infty} \Gamma_n (1-\kappa)^n. \tag{12}$$

The weak coupling expansion (11) diverges almost as strongly as the weak coupling expansion (3) [8, 11, 17]. However, the strong coupling expansion (12) has some very useful properties [12, 18–20].

For the ground and first excited states of the quartic anharmonic oscillator, we numerically computed 200 coefficients Γ_n [18]. From these data, we obtained the large-order formula for the Γ_n coefficients of the quartic oscillator

$$\Gamma_n = -\frac{12^K}{K!}\frac{4\sqrt{6}}{\pi e^2}(2n)^{(K-1)/2}\mathrm{e}^{-2\sqrt{2n}} \tag{13}$$

where $K = 0, 1, 2, \ldots$ is the index of excitation.

This formula was generalized to an arbitrary m and B_m in [19]

$$\Gamma_n = -\frac{2^{K+3/2}a^{K+1/2}b}{\pi K!\sqrt{m}}(bn)^{(\frac{K+1}{m}-1)}\mathrm{e}^{-[\frac{m}{m-1}(bn)^{1/m}]} \quad m \geqslant 3 \tag{14}$$

where

$$a = \frac{\Gamma(2m/(m-1))}{\Gamma^2(m/(m-1))} \tag{15}$$

and

$$b = 4(m-1)B_m a^{1-m}. \tag{16}$$

For $m = 2$, the right-hand side of equation (14) has to be divided by e^2. For $m = 2, 3, 4, 5$ and $K = 0, 1$, we compared equation (14) with the computed coefficients Γ_n and performed the

asymptotic analysis of the ratio of the numerical values of Γ_n and the values following from equation (14) (see [19]). The large-order formula (14) was also used in the summation rule

$$\Sigma_0 = \sum_{n=0}^{\infty} \Gamma_n = 2K + 1 \tag{17}$$

valid for arbitrary $m = 2, 3, \ldots$. It was shown in [19] that: (1) Equation (14) can be used at least qualitatively from n of the order of ten; (2) the absolute value of equation (14) is an upper bound to the absolute values of the actual Γ_n coefficients; (3) the asymptotic behaviour of the numerical coefficients Γ_n is given by equation (14); (4) the use of equation (14) in the summation rule (17) improves its accuracy, (5) finally, the results of [12, 18–20] show that, in contrast to the strong coupling expansion (4), the renormalized strong coupling expansion (12) converges for arbitrary $m \geqslant 2$, $K \geqslant 0$ and $\kappa \in (0, 2)$.

The large-order behaviour of the K_n coefficients was investigated in [21], where the large-order formula for the K_n coefficients

$$K_n = A\frac{\cos(n\varphi + \delta)}{|z_K|^n n^{3/2}} \tag{18}$$

where $\varphi = \arg z_K$ was derived. Here, A and δ are constants, z_K denotes the complex square-root branch point of the energy $\epsilon(z)$ of a given state K with the smallest distance to the origin [1–3, 22]

$$\epsilon(z) = \beta^{-1/(m+1)}E(\beta) = \sum_{n=0}^{\infty} K_n z^n \tag{19}$$

and $z = \beta^{-2/(m+1)}$. The values of $z_0 = z_2$ and z_4 of the quartic oscillator are known from [23]. The value of the branch point z_0 is also known for the sextic, octic and decadic oscillators [24]. The importance of the branch point z_K follows from the fact that it determines the minimal value of β for which the series (4) converges. It follows from equations (4) and (18) that

$$\beta_{min} = \frac{1}{|z_K|^{(m+1)/2}}. \tag{20}$$

It was shown in [24] that the general large-order formula for the K_n coefficients reads

$$K_n = \frac{1}{|z_K|^{n-1}}[c_1 C_n^{(-1/2)}(\cos\varphi) + c_2|z_K|^2 C_n^{(-3/2)}(\cos\varphi) + \cdots] \tag{21}$$

where $C_n^{(\alpha)}(x)$ are the Gegenbauer polynomials. Further, it was shown in [24] that equation (21) can also be written in the form

$$K_n = \frac{1}{|z_K|^{n-1}n^{3/2}}[(e_1 + e_2/n + e_3/n^2 + \cdots)\cos(n\varphi) + (f_1 + f_2/n + f_3/n^2 + \cdots)\sin(n\varphi)] \tag{22}$$

where e_i and f_i are constants. Taking only the leading term proportional to $1/n^{3/2}$, this equation yields (18) with $A = |z_K|\sqrt{e_1^2 + f_1^2}$ and $\cos\delta = e_1/\sqrt{e_1^2 + f_1^2}$. A general accurate method of calculating the value of the branch point z_K from the numerical values of the K_n coefficients was suggested in [24].

The main purpose of this paper is to perform detailed numerical analysis of the convergent strong coupling expansions (4) and (12) and the corresponding wavefunctions which have not been investigated until now. First, we describe the numerical method used for computing the expansion coefficients (section 2). The large-order behaviour of the renormalized strong coupling coefficients Γ_n, the radii of convergence of the series (12), the sign pattern of the coefficients, the summation rules for the coefficients and the perturbation wavefunctions are

discussed in section 3. In section 4, we discuss the large-order behaviour of the ordinary strong coupling expansion coefficients K_n, calculate z_K and β_{min} for the final excited state and investigate the perturbation wavefunctions. In the conclusion, both the strong coupling perturbative approaches are compared.

2. Numerical method

We assume the usual formulation of the perturbation theory

$$H\psi = E\psi \tag{23}$$

where

$$H = H_0 + \lambda H_1 \tag{24}$$

$$\psi = \psi_0 + \lambda\psi_1 + \lambda^2\psi_2 + \cdots \tag{25}$$

and

$$E = E_0 + \lambda E_1 + \lambda^2 E_2 + \cdots. \tag{26}$$

The well known equations for the perturbation energies E_n and wavefunctions ψ_n read

$$H_0\psi_0 = E_0\psi_0 \tag{27}$$

and

$$H_0\psi_n + H_1\psi_{n-1} = \sum_{i=0}^{n} E_i\psi_{n-i} \qquad n = 1, 2, \ldots. \tag{28}$$

Our numerical method was formulated in [25,26] and commented in [27–30]. We assume that equations (27) and (28) are numerically integrated from the point $-x_0$ to the point x_0 with the boundary conditions

$$\psi_n(-x_0) = d \qquad \psi_n(x_0) = 0 \qquad n = 0, 1, \ldots. \tag{29}$$

Here, x_0 is a sufficiently large number and d is a constant different from zero. We also assume that the wavefunctions ψ_i and perturbation corrections E_i are already calculated for $i = 0, \ldots, n-1$. Then, it follows from equation (28) that the wavefunction ψ_n depends not only on the coordinate x but also on the perturbation energy E_n taken as a parameter

$$\psi_n = \psi_n(E_n, x). \tag{30}$$

It was shown in [25,26] that the function $\psi_n(E_n, x_0)$ is a linear function of the parameter E_n

$$\psi_n(E_n, x_0) = \psi_n(E_n', x_0) + (E_n - E_n')F(x_0). \tag{31}$$

Taking $E_n' = 0$ and assuming in agreement with equation (29) $\psi_n(E_n, x_0) = 0$ we get the equation for the sought value of E_n for which the boundary condition $\psi_n(x_0) = 0$ is obeyed:

$$E_n = \frac{-\psi_n(0, x_0)}{F(x_0)} \qquad n \geqslant 1. \tag{32}$$

To compute E_n, only the values of the functions $\psi_n(0, x_0)$ and $F(x_0)$ are needed. The value of $F(x_0)$, which is independent of n, can easily be calculated from equation (31) for two arbitrarily chosen values of E_n and E_n', $E_n \neq E_n'$ and $n = 1$.

Equation (28) can be solved with the usual orthogonalization condition (see our discussion in [26]):

$$\langle\psi_0|\psi_n\rangle = \delta_{n0}. \tag{33}$$

However, it is obvious that this modification of the algorithm used in [30] has no effect on the values of the energies E_n. Thus, as far as the perturbation energies are concerned, the use of equation (33) is not necessary and only increases the computational time. As we show in the following section, the application of equation (33) may not be convenient even from the point of view of the wavefunctions when simple physical interpretation of the wavefunctions ψ_n can be lost.

The renormalized strong coupling case is obtained for $H_0 = p^2 + x^{2m}/B_m$, $H_1 = x^2 - x^{2m}/B_m$ and $\lambda = 1 - \kappa$. Analogously, the ordinary strong coupling case is obtained for $H_0 = p^2 + x^{2m}$, $H_1 = x^2$ and $\lambda = \beta^{-2/(m+1)}$.

3. Renormalized strong coupling case

3.1. Coefficients Γ_n

3.1.1. Ground and first excited states. Calculation of the coefficients Γ_n is difficult and, except for the quartic oscillator [18], only a limited number of these coefficients has been computed [12, 19]. Using the method described in section 2, we are able to compute a large number of the Γ_n coefficients for an arbitrary oscillator with the Hamiltonian (9). Since the quartic case was investigated in detail in [18] we limit ourselves to the discussion of the coefficients Γ_n for the ground and first excited states ($K = 0, 1$) of the sextic, octic and decadic oscillators ($m = 3, 4, 5$).

According to our experience, the usual computational accuracy does not yield reliable results at large orders of the perturbation theory (n about 100 and larger). For this reason, we used the language Maple with an adjustable number of decimal digits. We used 100 digits accuracy for the sextic oscillator, 125 digits for the octic oscillator and 175 digits for the decadic oscillator. The values of x_0 were taken $x_0 = 6.2$ for the sextic oscillator, $x_0 = 5.9$ for the octic oscillator and $x_0 = 5.4$ for the decadic oscillator. The resulting accuracy of the Γ_n coefficients is at least 50 digits which is necessary for the reliable computation of the K_n coefficients from the Γ_n coefficients (see section 4.1.1).

Selected coefficients Γ_n for the ground and first excited states of the sextic, octic and decadic oscillators are shown in tables 1 and 2. Similarly to the Γ_n coefficients for the quartic oscillator [18], the first two coefficients for the ground state of the sextic, octic and decadic oscillators are positive. However, in contrast to the quartic oscillator, the ground state coefficient Γ_3 is positive for these oscillators. In case of the first excited state, the sign pattern of the Γ_n coefficients for the sextic, octic and decadic oscillators is the same as for the quartic oscillator. We note that beginning with $n = 4$ for the ground state and $n = 2$ for the first excited state, all the coefficients Γ_n are negative in agreement with the large-order formula (14).

The accuracy of the computed coefficients Γ_n can be tested by means of the summation rules [12, 18]

$$\Sigma_j = \sum_{n=0}^{\infty}[n(n-1)\dots(n-j+1)\Gamma_n] = (-1)^j j! c_j. \tag{34}$$

From the numerically computed coefficients Γ_n, we can calculate only the partial sums

$$\Sigma_j^{(N)} = \sum_{n=0}^{N}[n(n-1)\dots(n-j+1)\Gamma_n]. \tag{35}$$

In tables 3 and 4, the values of $\Sigma_j^{(N)}$ for $N = 125$ and $j = 0, 1$ are compared with the exact

Table 1. Selected values of the coefficients Γ_n for the ground state ($K = 0$) of the sextic, octic and decadic oscillators ($m = 3, 4, 5$).

n	$m = 3$ Γ_n	$m = 4$ Γ_n	$m = 5$ Γ_n
0	0.625 089 812 5	0.555 130 236 0	0.503 318 917 6
1	0.407 659 180 6	0.500 898 761 7	0.578 406 995 4
2	−0.315 164 767 9e−1	−0.580 375 339 6e−1	−0.906 669 518 2e−1
3	0.642 929 680 7e−3	0.457 566 207 1e−2	0.120 372 764 9e−1
4	−0.553 405 367 0e−3	−0.571 262 654 0e−3	−0.794 449 938 7e−3
5	−0.392 921 718 6e−3	−0.525 828 977 1e−3	−0.769 618 471 6e−3
6	−0.244 097 416 7e−3	−0.272 173 161 1e−3	−0.145 371 012 2e−3
7	−0.167 766 666 7e−3	−0.214 187 016 7e−3	−0.200 337 885 7e−3
8	−0.118 767 916 7e−3	−0.164 649 523 0e−3	−0.163 584 063 0e−3
9	−0.862 006 277 6e−4	−0.127 557 016 5e−3	−0.125 036 986 4e−3
10	−0.639 829 952 0e−4	−0.101 300 583 5e−3	−0.104 024 507 3e−3
20	−0.672 691 832 5e−5	−0.184 812 869 8e−4	−0.246 962 137 9e−4
40	−0.397 936 144 4e−6	−0.243 467 474 1e−5	−0.466 262 441 7e−5
60	−0.555 234 420 3e−7	−0.631 948 747 9e−6	−0.158 140 803 8e−5
80	−0.116 764 842 1e−7	−0.224 282 875 1e−6	−0.699 250 213 8e−6
100	−0.314 767 049 4e−8	−0.957 380 574 6e−7	−0.360 768 867 2e−6
125	−0.771 174 119 0e−9	−0.391 152 113 0e−7	−0.181 382 482 0e−6

Table 2. Selected values of the coefficients Γ_n for the first excited state ($K = 1$) of the sextic, octic and decadic oscillators ($m = 3, 4, 5$).

n	$m = 3$ Γ_n	$m = 4$ Γ_n	$m = 5$ Γ_n
0	2.368 979 771 4	2.153 766 002 1	1.975 493 814 8
1	0.723 118 311 0	0.954 429 512 5	1.146 380 631 9
2	−0.431 081 529 1e−1	−0.540 686 823 1e−1	−0.723 953 396 3e−1
3	−0.158 281 475 7e−1	−0.137 859 664 1e−1	−0.107 716 739 0e−1
4	−0.934 323 352 3e−2	−0.859 311 646 6e−2	−0.654 430 077 2e−2
5	−0.604 878 249 5e−2	−0.606 039 306 7e−2	−0.499 939 393 6e−2
6	−0.412 813 241 5e−2	−0.443 889 685 3e−2	−0.381 394 463 1e−2
7	−0.293 301 664 8e−2	−0.336 291 435 1e−2	−0.298 478 118 8e−2
8	−0.215 043 421 3e−2	−0.261 847 789 1e−2	−0.239 580 073 3e−2
9	−0.161 687 390 9e−2	−0.208 414 485 0e−2	−0.196 186 518 2e−2
10	−0.124 108 209 6e−2	−0.168 904 987 3e−2	−0.163 292 955 4e−2
20	−0.169 825 905 6e−3	−0.365 382 822 9e−3	−0.437 333 455 4e−3
40	−0.136 622 790 4e−4	−0.592 826 908 4e−4	−0.962 701 857 7e−4
60	−0.228 871 000 8e−5	−0.175 077 133 5e−4	−0.360 010 321 4e−4
80	−0.546 009 067 5e−6	−0.681 095 858 8e−5	−0.170 844 612 3e−4
100	−0.161 803 306 7e−6	−0.311 956 987 8e−5	−0.931 290 235 4e−5
125	−0.434 440 450 8e−7	−0.136 611 765 9e−5	−0.494 604 037 4e−5

values of the sums Σ_0 and Σ_1 given by equation (17) and by the equation

$$\Sigma_1 = \sum_{n=0}^{\infty} n\Gamma_n = -c_1. \tag{36}$$

For $K = 0$ and $K = 1$, the coefficient c_1 is given by the equation

$$c_1 = \frac{\langle\varphi_0|x^{2m}/B_m - x^2|\varphi_0\rangle}{\langle\varphi_0|\varphi_0\rangle} = -\frac{m-1}{2m} \tag{37}$$

Table 3. Summation rules for the coefficients Γ_n for the ground state of the quartic, sextic, octic and decadic oscillators ($m = 2, 3, 4, 5$). Σ_j is the exact value of the summation rule for the infinite number of terms, $\Sigma_j^{(N)}$ denotes the partial sum for $n = 0, \ldots, N$ and Σ_j^{LO} equals $\Sigma_j^{(N)}$ plus the rest of the sum in which the large-order formula (14) for $n = N + 1, \ldots, 5000$ was used. Here, $N = 125$.

		$m = 2$		$m = 3$		$m = 4$		$m = 5$	
j	Σ_j	$\Sigma_j^{(N)} - \Sigma_j$	$\Sigma_j^{LO} - \Sigma_j$	$\Sigma_j^{(N)} - \Sigma_j$	$\Sigma_j^{LO} - \Sigma_j$	$\Sigma_j^{(N)} - \Sigma_j$	$\Sigma_j^{LO} - \Sigma_j$	$\Sigma_j^{(N)} - \Sigma_j$	$\Sigma_j^{LO} - \Sigma_j$
0	1	0.339e−14	−0.265e−15	0.160e−7	−0.805e−8	0.142e−5	−0.298e−6	0.951e−5	−0.139e−5
1	$\frac{m-1}{2m}$	0.453e−12	−0.354e−13	0.239e−5	−0.119e−5	0.245e−3	−0.493e−4	0.191e−2	−0.258e−3

Table 4. Summation rules for the coefficients Γ_n for the first excited state of the quartic, sextic, octic and decadic oscillators ($m = 2, 3, 4, 5$). Σ_j is the exact value of the summation rule for the infinite number of terms, $\Sigma_j^{(N)}$ denotes the partial sum for $n = 0, \ldots, N$ and Σ_j^{LO} equals $\Sigma_j^{(N)}$ plus the rest of the sum in which the large-order formula (14) for $n = N + 1, \ldots, 5000$ was used. Here, $N = 125$.

		$m = 2$		$m = 3$		$m = 4$		$m = 5$	
j	Σ_j	$\Sigma_j^{(N)} - \Sigma_j$	$\Sigma_j^{LO} - \Sigma_j$	$\Sigma_j^{(N)} - \Sigma_j$	$\Sigma_j^{LO} - \Sigma_j$	$\Sigma_j^{(N)} - \Sigma_j$	$\Sigma_j^{LO} - \Sigma_j$	$\Sigma_j^{(N)} - \Sigma_j$	$\Sigma_j^{LO} - \Sigma_j$
0	1	0.582e − 12	−0.134e − 12	0.965e − 6	−0.592e − 6	0.543e − 4	−0.183e − 4	0.286e − 3	−0.849e − 4
1	$\frac{m-1}{2m}$	0.780e − 10	−0.180e − 10	0.145e − 3	−0.881e − 4	0.958e − 2	−0.308e − 2	0.603e − 1	−0.164e − 1

where $|\varphi_0\rangle$ is the unperturbed ground or first excited state wavefunction of the harmonic oscillator. It is seen from tables 3 and 4 that the difference $\Sigma_j^{(N)} - \Sigma_j$ is always positive in agreement with the negative sign of the coefficients Γ_n for large n. The agreement of $\Sigma_j^{(N)}$ and Σ_j is excellent for the ground state of the quartic oscillator ($K = 0$ and $m = 2$). This agreement goes down with increasing m and K as can be seen from equation (14) which shows that the series (12) converges less rapidly with increasing m and K. Comparing the $j = 0$ and $j = 1$ results we see that the $j = 0$ case leads to better agreement than the $j = 1$ case. Again, this result can be expected because of the increased contribution of the terms with large n in equation (36) in comparison with equation (17). Similarly to [18, 19], we also calculated the sums

$$\Sigma_j^{LO} = \sum_{n=0}^{N} [n(n-1)\ldots(n-j+1)\Gamma_n] + \sum_{n=N+1}^{\infty} [n(n-1)\ldots(n-j+1)\Gamma_n] \tag{38}$$

where we used the numerical values of the Γ_n coefficients in the first sum and the large-order formula (14) in the second sum. The infinite upper bound in the second sum was replaced by 5000. It is seen that if the second sum is included into the summation rule its accuracy increases about one order. At the same time, the difference $\Sigma_j^{LO} - \Sigma_j$ is negative which shows that the absolute value of the large-order formula (14) is an upper bound to the actual values of the coefficients Γ_n.

It follows from this discussion that truncating the series (12) at some large n, we get an upper bound to the exact energy $E_R(\kappa)$. If we use the large-order formula (14) in the remainder of the series and add it to the truncated series we get the lower bound.

We also calculated the radius of convergence r of the series (12) (see also [19]). The ratio $|\Gamma_{n-1}/\Gamma_n|$ appearing in the d'Alembert convergence criterion was computed for the ground and first excited states of the quartic, sextic, octic and decadic oscillators for $n = 109, \ldots, 125$. These values were extrapolated to $n \to \infty$ by means of the Thiele extrapolation built in Maple (see table 5). In the extrapolation, the variable $1/n^{1/m}$ appearing in equation (14) was used. It is

Table 5. The radius of convergence of the renormalized strong coupling expansion (12) obtained from the d'Alembert convergence criterion for the ground and first excited states ($K = 0, 1$) of the quartic, sextic, octic and decadic oscillators ($m = 2, 3, 4, 5$).

	$m = 2$	$m = 3$	$m = 4$	$m = 5$
K	r	r	r	r
0	1.000 006	1.000 02	0.999 98	0.9998
1	1.000 02	1.000 03	1.001	0.9995

seen from table 5 that this direct computation yields the values of r which are very close to one. This test independent of the large-order formula (14) confirms that the renormalized strong coupling expansion (12) converges for $\kappa \in (0, 2)$. It agrees with our conclusions in [18–20].

3.1.2. Excited states of the quartic oscillator. We investigated the coefficients Γ_n for higher excited states of the quartic oscillator for $K = 2, \ldots, 10$ (see table 6). It is seen that the coefficients Γ_n are, except for the $n = 0$, $K = 2, \ldots, 10$ and $n = 1$, $K = 2$ cases, negative. With increasing n, their behaviour can approximately be described by equation (14). However, the relative difference of the numerical values of Γ_n and equation (14) increases rapidly with increasing K. To achieve better agreement of the values of Γ_n and equation (14) it would be necessary to take into account corrections to the leading term as it was done in [18, 19] or to go to very large n. The absolute value of the Γ_n coefficients for $n = 100$ is still relatively large. To get lower absolute values of the coefficients for large n it would be necessary to take another value of the constant $B_m = 3$ which was optimized for the ground state [10].

We also verified that these coefficients Γ_n obey the summation rule (17) and that the expansion (12) converges for $\kappa \in (0, 2)$ (see also [19, 20]).

3.2. Wavefunctions ψ_n

To the best of our knowledge, the wavefunctions of the anharmonic oscillator were investigated in [31] only, where the ground state wavefunctions for the quartic oscillator with the Hamiltonian H were investigated by means of the optimized δ expansion. For this reason, we decided to perform detailed analysis of the wavefunctions here. Our approach is more straightforward and transparent than that used in [31].

First we note that the form of the wavefunctions ψ_n depends on the form of the initial conditions and on the fact of whether or not the orthogonality condition (33) is applied. We found that the functions ψ_n have a simpler form with a straightforward interpretation if this condition is not applied.

In our calculations, we used the following initial conditions for the unperturbed wavefunction ψ_0:

$$\psi_0(x = 0) = 1 \qquad \mathrm{d}\psi_0/\mathrm{d}x(x = 0) = 0 \tag{39}$$

for even parity states ($K = 0, 2, \ldots$) and

$$\psi_0(x = 0) = 0 \qquad \mathrm{d}\psi_0/\mathrm{d}x(x = 0) = 1 \tag{40}$$

for odd parity states ($K = 1, 3, \ldots$). For the perturbation corrections ψ_n to ψ_0 we used the conditions

$$\psi_n(x = 0) = 0 \qquad \mathrm{d}\psi_n/\mathrm{d}x(x = 0) = 0 \qquad n \geqslant 1. \tag{41}$$

The wavefunctions ψ_n for the ground state of the quartic oscillator are shown in figure 1. Because of the symmetry of the wavefunctions, only the $x > 0$ part of the functions is shown.

Neither the functions ψ_n nor the resulting wavefunction ψ are normalized. The first few overlap integrals $S_{0n} = \langle\psi_0|\psi_n\rangle$ for the ground state equal $S_{00} = 1.13$, $S_{01} = -0.046$, $S_{02} = 0.0038$, $S_{03} = 0.000\,31$, $S_{04} = 0.000\,18$, $S_{05} = 0.000\,092$. We see that the functions ψ_n are 'almost orthogonal' and the overlap integrals are, except for S_{01}, positive.

It is seen from figure 1 that the perturbation series for the renormalized wavefunction

$$\psi_R = \sum_{n=0}^{\infty}(1-\kappa)^n\psi_n \tag{42}$$

has a very simple physical interpretation. The function ψ_0 corresponds to the Hamiltonian H_0, where $H_0 = p^2 + x^4/3$. The wavefunction ψ_R corresponding to the Hamiltonian $H_R = H_0 + (1-\kappa)H_1$, where $H_1 = x^2 - x^4/3$ has to decay less slowly than the function ψ_0. Therefore, the function ψ_1 is first negative and then positive. Then, the function ψ_2 corrects the behaviour of the function ψ_0 in more detail. It is seen from figures 1(*a*) and (*b*) that the ground state function ψ_2 has a minimum at the point where ψ_1 changes its sign. Beginning from $n \geqslant 3$ (see figures 1(*b*)–(*d*)), the functions ψ_n have very simple form. They are positive for all x, their maximum shifts with increasing n to larger values of x and the value of their maximum goes down. With increasing n, smaller corrections $(1-\kappa)^n\psi_n$ to ψ_R in the region more distant from the origin are obtained. Therefore, truncating the perturbation series (42) one can make the error caused by the truncation arbitrarily small.

From the form of the functions ψ_n, we can understand the signs of the Γ_n coefficients for the ground state of the quartic oscillator.

The function ψ_0 shown in figure 1(*a*) is the ground state solution of the Schrödinger equation with the Hamiltonian H_0. The corresponding eigenvalue $\Gamma_0 = 0.735\,21$ [18] must lie above the minimum of the potential and, therefore, must be positive.

The first energy correction is given by the well known equation

$$\Gamma_1 = \frac{W_{00}}{S_{00}} \tag{43}$$

where $W_{0n} = \langle\psi_0|H_1|\psi_n\rangle$. Taking into account that the potential H_1 is positive for $x \in (0, \sqrt{3})$ and the function ψ_0 decays rapidly for $x > \sqrt{3}$ we see that the coefficient Γ_1 has to be positive in agreement with $\Gamma_1 = 0.277\,05$ [18].

Further coefficients Γ_n are given by the equation following from equation (28):

$$\Gamma_n = \frac{W_{0,n-1} - \sum_{i=1}^{n-1}\Gamma_i S_{0,n-i}}{S_{00}}. \tag{44}$$

For $n = 2$, the function ψ_1 is negative for small $x > 0$ (see figure 1(*b*)) so that $W_{01} = -0.025$ is also negative. Taking into account the values of W_{01}, S_{00}, S_{01} and Γ_1 in equation (44) for $n = 2$ we get the negative value of Γ_2 in agreement with $\Gamma_2 = -0.011\,178$ [18].

Because of the form of the wavefunctions ψ_n (see figure 1(*b*)–(*d*) and the discussion above) and the form of the potential H_1 the matrix elements $W_{0,n-1}$ are positive for $n = 3$ and negative for $n > 3$. At the same time, the sum $\sum_{i=1}^{n-1}\Gamma_i S_{0,n-i}$ is positive for $n \geqslant 3$. Thus, the sign of the coefficient Γ_3 depends on the absolute values of these terms. It is negative for the ground and first excited states of the quartic oscillator and the first excited state of the sextic, octic and decadic oscillators (see [18] and table 2). However, it is positive for the ground state of the sextic, octic and decadic oscillators (see table 1).

For $n \geqslant 4$, the term $W_{0,n-1}$ is negative. It is consequence of the fact that the functions ψ_0 and ψ_n, $n \geqslant 2$ have positive values and the functions ψ_n, $n > 2$ have maximum in the region where the perturbation potential H_1 becomes negative (see above). Since both the terms $W_{0,n-1}$ and $-\sum_{i=1}^{n-1}\Gamma_i S_{0,n-i}$ are negative the coefficients Γ_n are for $n > 4$ negative.

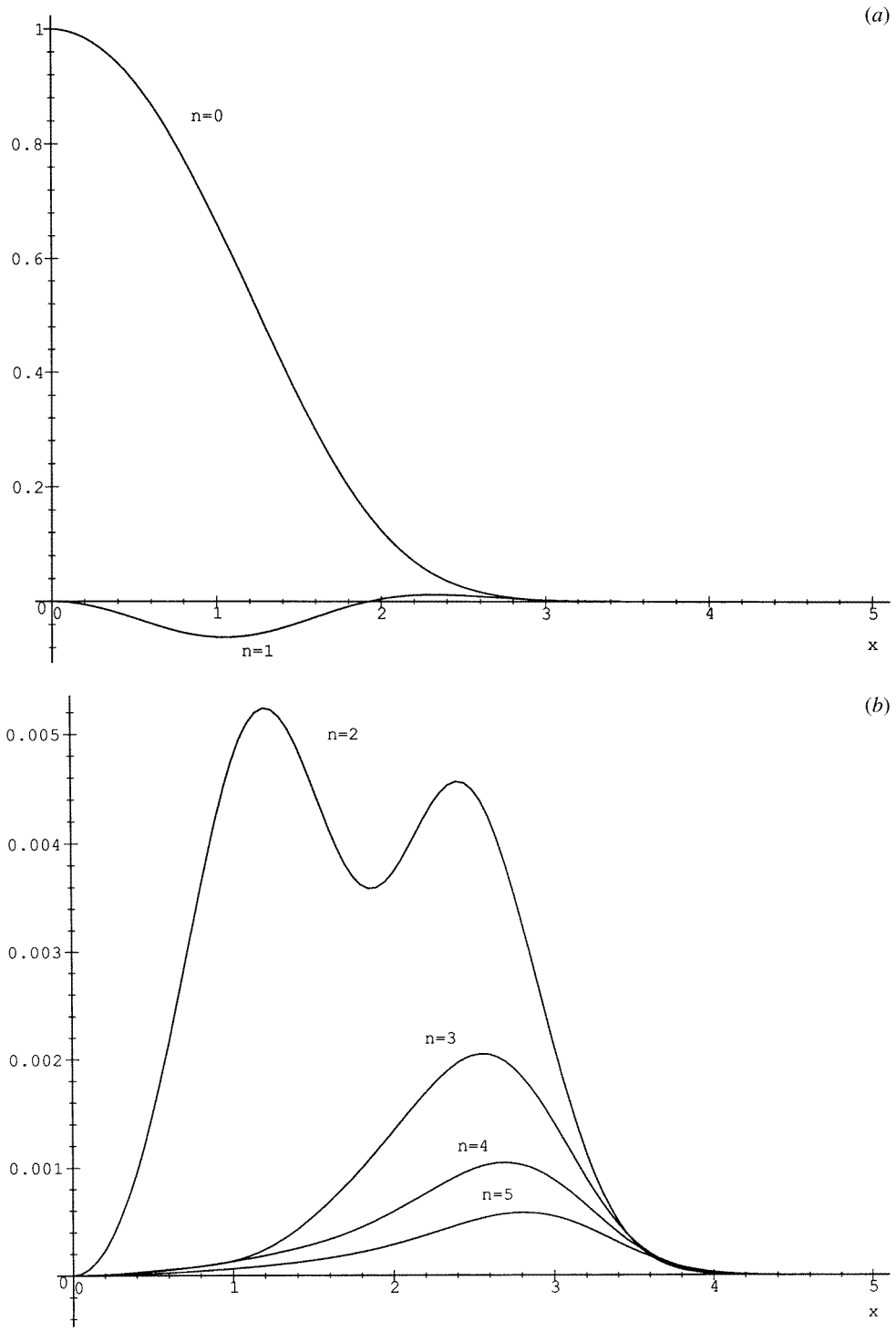

Figure 1. The perturbation wavefunctions ψ_n for the ground state of the quartic oscillator with the renormalized Hamiltonian $H_R = H_0 + (1-\kappa)H_1$, where $H_0 = p^2 + x^4/3$ and $H_1 = x^2 - x^4/3$. Because of the symmetry of the functions ψ_n, only the $x > 0$ part of the functions is shown. (*a*) $n = 0, 1$, (*b*) $n = 2, \ldots, 5$, (*c*) $n = 6, \ldots, 10$, (*d*) $n = 21, \ldots, 25$.

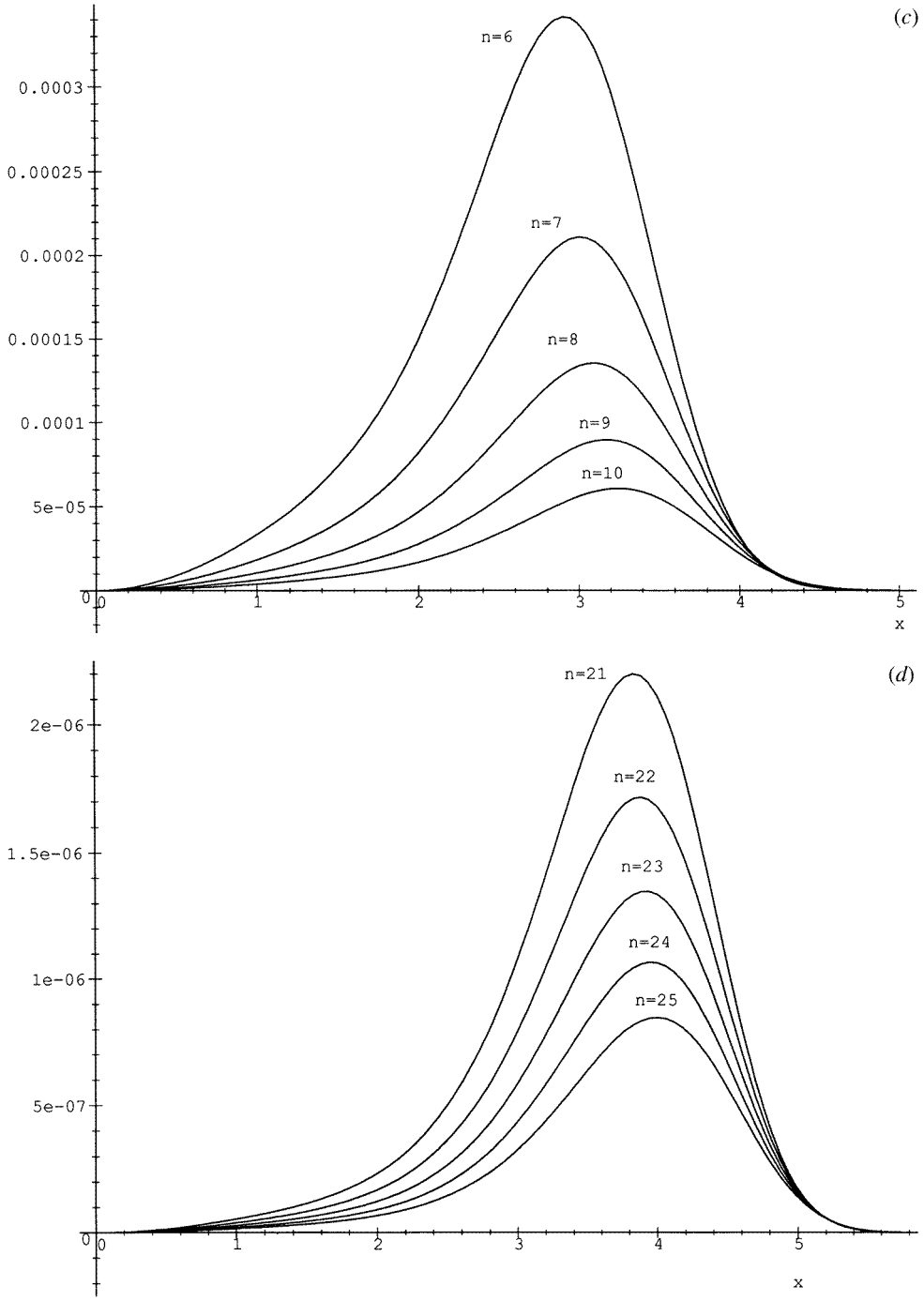

Figure 1. (Continued)

The main arguments of this discussion also apply for the excited states and for the higher-order oscillators with $m = 3, 4, 5$ where the perturbation potential $H_1 = x^2 - x^{2m}/B_m$ and the wavefunctions ψ_n have analogous form.

4. Ordinary strong coupling case

4.1. Coefficients K_n

4.1.1. Ground and first excited states. The coefficients K_n can be computed by the method described in section 2. They can also be computed from the Γ_n coefficients via the equations [32]

$$K_0 B_m^{\frac{-1}{m+1}} = \Gamma_0 \tag{45}$$

and

$$K_n B_m^{\frac{2n-1}{m+1}} = \Gamma_n - \sum_{i=0}^{n-1} K_i \frac{B_m^{\frac{2i-1}{m+1}}}{(n-i)!} \frac{\Gamma\left(\frac{2i-1}{m+1} + n - i\right)}{\Gamma\left(\frac{2i-1}{m+1}\right)} \tag{46}$$

following from the comparison of the series (4) and (12). We note that the coefficient K_n depends on the coefficients $\Gamma_0, \ldots, \Gamma_n$ and vice versa. With increasing n, the K_n coefficients go to zero more quickly than the Γ_n coefficients and cancellation of large terms in equation (46) requires very high computational accuracy. In our calculations, we used 250 decimal digits accuracy. The accuracy of the energy $E(\beta)$ given by equation (4) is similar to that achieved in [33, 34] by means of the numerical integration of the Bloch equation (about 70 digits for $\beta = 1$).

Until now, only a small number of the K_n coefficients have been computed (see e.g. [12, 21, 30, 32, 35–38]). For this reason, we calculated the K_n coefficients for the ground and first excited states of the quartic, sextic, octic and decadic oscillators (see tables 7 and 8). We note that the absolute values of these coefficients go down with increasing n more rapidly than in case of the Γ_n coefficients (cf tables 7 and 8 with tables 1 and 2 and [18]). With increasing m and K, the coefficients K_n go down more quickly than for the ground state of the quartic oscillator ($m = 2$ and $K = 0$). It is seen that about 20 first terms of the series (4) are sufficient to achieve very high accuracy of the energy (4) for $\beta \geqslant 1$.

The large-order analysis of the ground state K_n coefficients (table 7) was performed in [24]. Here, also we perform a similar analysis for the first excited state (table 8). The corresponding values of the branch points z_0, β_{min} and the constants $c_1, \ldots, c_4$ in equation (21) are given in table 9. It is seen that the values of the coefficients c_i go down with increasing i so that our restriction to a few terms in the expansion (21) is justified. The coefficients c_i depend slightly on n_0 so that their values are less reliable than the values of z_K and β_{min}. Because of the prefactor $\beta^{1/(m+1)}$ in equation (4) which goes to zero for $\beta \to 0$, the expansion $\sum_n K_n \beta^{-2n/(m+1)}$ must diverge for $\beta \to 0$ when $E(0) = 2K + 1$. It is seen from table 9 that β_{min} goes down with increasing m in agreement with the behaviour of the prefactor $\beta^{1/(m+1)}$ which goes to zero more slowly if m is increased.

4.1.2. Excited states of the quartic oscillator. In contrast to the Γ_n coefficients, dependence of the coefficients K_n on m and K is more complex. To clarify their K-dependence, we computed 100 K_n coefficients for higher excited states ($K = 2, \ldots, 10$) of the quartic oscillator (see table 10). It is seen that the coefficients K_n for large K go down with increasing n more rapidly than the coefficients Γ_n shown in table 6.

Table 6. Selected values of the coefficients Γ_n for the excited states ($K = 2, \ldots, 10$) of the quartic oscillator.

n	$K=2$ Γ_n	$K=3$ Γ_n	$K=4$ Γ_n	$K=5$ Γ_n	$K=6$ Γ_n	$K=7$ Γ_n	$K=8$ Γ_n	$K=9$ Γ_n	$K=10$ Γ_n
0	5.169 4	8.074 0	11.275	14.725	18.393	22.255	26.294	30.494	34.845
1	0.720 24e−1	−0.444 44	−1.102 3	−1.872 6	−2.737 8	−3.685 6	−4.707 0	−5.794 9	−6.943 7
2	−0.115 79	−0.289 47	−0.508 20	−0.764 76	−1.053 0	−1.3688	−1.7093	−2.0719	−2.4548
3	−0.539 94e−1	−0.133 01	−0.240 40	−0.369 51	−0.516 93	−0.680 23	−0.857 64	−1.0478	−1.2495
4	−0.275 82e−1	−0.717 89e−1	−0.134 59	−0.212 19	−0.302 30	−0.403 29	−0.513 95	−0.633 33	−0.760 69
5	−0.155 73e−1	−0.427 06e−1	−0.830 97e−1	−0.134 39	−0.194 99	−0.263 70	−0.339 66	−0.422 15	−0.510 62
6	−0.939 38e−2	−0.270 93e−1	−0.546 63e−1	−0.906 49e−1	−0.133 89	−0.183 52	−0.238 86	−0.299 37	−0.364 60
7	−0.594 18e−2	−0.179 92e−1	−0.375 97e−1	−0.638 89e−1	−0.960 37e−1	−0.133 37	−0.175 37	−0.221 60	−0.271 71
8	−0.389 66e−2	−0.123 64e−1	−0.267 30e−1	−0.465 19e−1	−0.711 37e−1	−0.100 07	−0.132 90	−0.169 29	−0.208 94
9	−0.262 91e−2	−0.872 72e−2	−0.194 99e−1	−0.347 33e−1	−0.540 14e−1	−0.769 51e−1	−0.103 20	−0.132 50	−0.164 59
10	−0.181 54e−2	−0.629 39e−2	−0.145 19e−1	−0.264 58e−1	−0.418 30e−1	−0.603 36e−1	−0.817 08e−1	−0.105 71	−0.132 15
20	−0.921 68e−4	−0.463 54e−3	−0.141 46e−2	−0.316 77e−2	−0.583 21e−2	−0.944 19e−2	−0.139 92e−1	−0.194 57e−1	−0.258 05e−1
40	−0.119 07e−5	−0.104 38e−4	−0.493 84e−4	−0.155 47e−3	−0.372 29e−3	−0.739 38e−3	−0.128 78e−2	−0.203 99e−2	−0.301 09e−2
60	−0.361 24e−7	−0.480 72e−6	−0.323 92e−5	−0.135 80e−4	−0.409 14e−4	−0.976 36e−4	−0.197 16e−3	−0.352 33e−3	−0.574 54e−3
80	−0.171 98e−8	−0.317 14e−7	−0.288 35e−6	−0.155 52e−5	−0.576 34e−5	−0.162 90e−4	−0.377 85e−4	−0.756 92e−4	−0.135 74e−3
100	−0.110 73e−9	−0.264 57e−8	−0.311 34e−7	−0.210 49e−6	−0.943 30e−6	−0.312 26e−5	−0.825 76e−5	−0.184 50e−4	−0.362 60e−4

Table 7. Selected values of the coefficients K_n for the ground state ($K = 0$) of the quartic, sextic, octic and decadic oscillators ($m = 2, 3, 4, 5$).

n	$m = 2$ K_n	$m = 3$ K_n	$m = 4$ K_n	$m = 5$ K_n
0	1.060 362 090 4	1.144 802 453 7	1.225 820 113 8	1.298 843 700 6
1	0.362 022 648 7	0.307 920 303 7	0.277 118 934 3	0.256 647 384 3
2	−0.345 102 627 2e−1	−0.185 416 643 1e−1	−0.126 322 842 6e−1	−0.966 539 385 8e−2
3	0.519 530 271 0e−2	0.155 974 219 5e−2	0.750 441 570 4e−3	0.454 884 856 8e−3
4	−0.830 834 446 3e−3	−0.123 901 174 3e−3	−0.385 978 159 5e−4	−0.174 023 066 4e−4
5	0.129 111 907 7e−3	0.797 194 882 5e−5	0.127 080 594 2e−5	0.320 604 093 1e−6
6	−0.184 894 634 4e−4	−0.267 672 848 9e−6	0.247 628 889 9e−7	0.207 577 396 1e−7
7	0.226 366 476 0e−5	−0.251 217 514 9e−7	−0.824 648 630 3e−8	−0.262 918 217 6e − 8
8	−0.188 772 014 8e−6	0.632 251 434 0e−8	0.752 942 635 0e−9	0.148 693 421 1e−9
9	−0.652 387 107 2e−8	−0.762 588 422 7e−9	−0.381 276 272 7e−10	−0.324 258 771 5e−11
10	0.777 550 922 9e−8	0.589 732 972 7e−10	0.440 371 083 8e−14	−0.280 139 814 3e−12
20	−0.728 030 338 0e−15	0.101 917 009 9e−19	−0.479 519 644 7e−22	−0.838 178 795 1e−24
40	0.753 983 426 9e−29	−0.459 445 426 1e−39	0.108 237 647 7e−43	0.534 929 336 7e−47
60	−0.119 652 484 8e−42	−0.886 045 027 3e−57	0.367 032 022 8e−65	−0.449 529 332 7e−70
80	0.220 371 222 4e−56	−0.476 581 002 1e−75	−0.112 061 465 0e−85	0.292 188 679 2e−93
100	−0.428 794 376 1e−70	−0.174 562 472 9e−93	0.130 117 803 9e−106	0.189 397 346 7e−116

Table 8. Selected values of the coefficients K_n for the first excited state ($K = 1$) of the quartic, sextic, octic and decadic oscillators ($m = 2, 3, 4, 5$).

n	$m = 2$ K_n	$m = 3$ K_n	$m = 4$ K_n	$m = 5$ K_n
0	3.799 673 029 8	4.338 598 711 5	4.755 874 413 9	5.097 876 529 2
1	0.901 605 895 8	0.718 220 132 3	0.627 299 876 8	0.571 825 765 7
2	−0.574 830 897 3e−1	−0.243 956 823 1e−1	−0.147 491 062 4e−1	−0.105 411 773 1e−1
3	0.549 274 610 2e−2	0.999 479 525 6e−3	0.355 337 961 4e−3	0.176 289 730 7e−3
4	−0.513 896 977 0e−3	−0.262 855 964 1e−4	−0.215 636 160 2e−5	0.452 972 396 5e−6
8	0.397 970 158 8e−4	−0.522 469 402 0e−6	−0.319 401 686 1e−6	−0.138 645 169 3e−6
6	−0.164 663 897 4e−5	0.110 672 835 1e−6	0.150 736 375 1e−7	0.302 726 736 7e−8
7	−0.179 706 613 5e−6	−0.594 143 419 4e−8	−0.228 150 685 9e−10	0.801 921 428 6e−10
8	0.559 964 386 1e−7	−0.441 005 734 7e−13	−0.292 862 695 9e−10	−0.614 766 142 6e−11
9	−0.817 574 433 4e−8	0.255 393 681 2e−10	0.130 247 317 3e−11	0.654 670 770 1e−13
10	0.733 621 976 6e−9	−0.200 288 119 9e−11	0.989 293 173 0e−14	0.709 658 594 3e−14
20	0.715 133 092 5e−18	0.803 323 203 7e−23	−0.101 536 175 8e−25	0.176 597 690 4e−28
40	−0.484 265 453 0e−34	0.366 914 884 4e−45	0.572 384 123 3e−51	−0.836 618 503 0e−55
60	−0.274 088 828 7e−50	0.199 679 344 8e−67	−0.506 457 559 7e−76	−0.192 095 977 7e−82
80	−0.480 006 056 6e−67	0.347 575 736 2e−90	0.537 053 919 1e−101	0.101 673 648 6e−108
100	0.412 350 251 3e−83	−0.200 998 745 8e−111	−0.628 371 240 1e−126	0.436 373 942 2e−136

We investigated also the large-order behaviour of the K_n coefficients. Our numerical calculations (see tables 7 and 10) show that the large-order coefficients K_n for $K = 0$ and $K = 2$ obey with a high accuracy the relation

$$K_n^{K=0} = -K_n^{K=2}. \tag{47}$$

We verified that, in agreement with [23], the values of the branch points z_K and β_{min} are the same for $K = 0$ and $K = 2$. At the same time, the coefficients $c_i^{K=2}$ in equation (21) for $K = 2$ obey the equation

$$c_i^{K=2} = -c_i^{K=0} \qquad i = 1, \ldots, 4. \tag{48}$$

Table 9. The square-root branch point z_1, β_{min} and the coefficients $c_1, \ldots, c_4$ in the large-order formula (21) describing the large-order behaviour of the coefficients K_n for the first excited state ($K = 1$) of the quartic, sextic, octic and decadic oscillators ($m = 2, 3, 4, 5$). The values of the coefficients c_i were calculated by the method described in [24] for $n = n_0$.

m	n_0	z_1	β_{min}	c_1	c_2	c_3	c_4
2	106	−4.987 283 7 + 4.023 154 3i	0.061 651 769	−0.796 90	0.144e−1	0.1e−3	0.5e−6
3	85	−7.029 103 9 + 10.021 258 3i	0.006 674 056 1	−0.507 007	0.7e−3	−0.1e−5	−0 .2e−8
4	88	−7.863 810 32 + 15.461 323 1i	0.000 797 970 16	−0.386 025	0.882e−4	0.1e−7	0.2e−9
5	85	−8.255 474 46 + 20.056 268 3i	0.000 098 014 351	−0.352 687	0.592e−4	0.5e−8	0.6e−10

For $K > 3$, the large-order behaviour of the K_n coefficients can be described by equations (18) or (21) only approximately. To suppress the contribution of the other branch points [23] it would be necessary to consider n much larger than 100 or to generalize equation (21) to a larger number of the branch points.

4.2. Wavefunctions ψ_n

The wavefunctions ψ_n for the ground state of the quartic oscillator corresponding to $H_0 = p^2 + x^4$, $H_1 = x^2$ and $\lambda = \beta^{-2/(m+1)}$ are shown in figure 2.

First we note that, in contrast to the functions ψ_n in the renormalized case (figure 1), the functions ψ_n in figure 2 change their sign. It is seen that maxima or minima of these functions shift with increasing n to larger values of x, however, this shift is less significant than in the renormalized case. Finally, we see that the norm of the functions ψ_n in figure 2 goes to zero with increasing n more rapidly than for the functions in figure 1.

Thus, the resulting situation is more complex than in the renormalized case. Terms in the equation for the K_n coefficients analogous to equation (44) have different signs and their cancellation can be expected. As a result, different signs of the K_n coefficients and rapidly decreasing absolute values of the coefficients with increasing n are obtained (see tables 7 and 8).

5. Conclusions

In this paper, we performed detailed numerical analysis of the convergent strong coupling expansions (4) and (12) of the energies $E(\beta)$ and $E_R(\kappa)$ for the anharmonic oscillators. Except for the expansion coefficients Γ_n and K_n, we also discussed the corresponding perturbation wavefunctions.

The ground and first excited states of the quartic, sextic, octic and decadic oscillators were investigated. The higher excited states of the quartic oscillator were also discussed.

For these cases, the expansion coefficients Γ_n and K_n, the large-order formulae for the coefficients, the radii of convergence of the perturbation series, the perturbation wavefunctions and the summation rules were investigated.

The properties of the renormalized strong coupling expansion (12) can be summarized as follows:

- This expansion converges for all $\kappa \in (0, 2)$; it means that it converges for all $\beta \in (0, \infty)$ corresponding to $\kappa \in (0, 1)$ and for the double-well potentials $V(x) = \kappa x^{2m}/B_m - (\kappa - 1)x^2$ in the renormalized Hamiltonian H_R corresponding to $\kappa \in (1, 2)$.
- The large-order formula (14) is analytic. It is at least qualitatively applicable from n of the order of ten.

Table 10. Selected values of the coefficients K_n for the excited states ($K = 2, \ldots, 10$) of the quartic oscillator.

n	$K=2$ K_n	$K=3$ K_n	$K=4$ K_n	$K=5$ K_n	$K=6$ K_n	$K=7$ K_n	$K=8$ K_n	$K=9$ K_n	$K=10$ K_n
0	7.455 7	11.644	16.261	21.238	26.528	32.098	37.923	43.981	50.256
1	1.244 7	1.557 9	1.841 6	2.105 0	2.352 8	2.588 3	2.813 4	3.029 9	3.239 0
2	−0.466 01e−1	−0.471 07e−1	−0.469 15e−1	−0.468 46e−1	−0.468 04e−1	−0.467 78e−1	−0.467 61e−1	−0.467 49e−1	−0.467 40e−1
3	0.958 84e−3	0.118 44e−2	0.978 79e−3	0.860 45e−3	0.770 56e−3	0.700 90e−3	0.645 03e−3	0.599 08e−3	0.560 50e−3
4	0.362 54e−3	0.868 33e−4	0.699 48e−4	0.509 33e−4	0.401 07e−4	0.327 76e−4	0.275 39e−4	0.236 24e−4	0.205 98e−4
5	−0.965 06e−4	−0.150 05e−4	−0.123 60e−4	−0.769 80e−5	−0.544 90e−5	−0.404 99e−5	−0.313 20e−5	−0.249 56e−5	−0.203 59e−5
6	0.164 38e−4	0.418 12e−6	0.116 274-5	0.557 26e−6	0.359 71e−6	0.242 32e−6	0.172 34e−6	0.127 45e−6	0.972 41e−7
7	−0.212 52e−5	0.228 26e−6	−0.105 18e−6	−0.262 03e−7	−0.172 25e−7	−0.103 64e−7	−0.679 04e−8	−0.466 03e−8	−0.332 51e−8
8	0.173 45e−6	−0.572 70e−7	0.139 79e−7	0.523 99e−9	0.846 41e−9	0.412 50e−9	0.250 40e−9	0.158 51e−9	0.105 45e−9
9	0.904 04e−8	0.813 18e−8	−0.240 89e−8	0.876 25e−10	−0.810 40e−10	−0.272 01e−10	−0.156 73e−10	−0.902 12e−11	−0.556 31e−11
10	−0.817 62e−8	−0.711 51e−9	0.386 94e−9	−0.259 82e−10	0.115 49e−10	0.258 18e−11	0.149 51e−11	0.780 92e−12	0.448 96e−12
20	0.728 35e−15	−0.736 77e−18	−0.327 85e−18	0.219 41e−19	−0.765 20e−21	−0.284 50e−21	0.129 12e−21	−0.189 15e−22	0.350 60e−23
40	−0.753 98e−29	0.484 29e−34	−0.510 71e−36	−0.261 32e−38	0.244 48e−40	0.543 55e−43	−0.432 27e−43	−0.602 25e−45	0.308 31e−46
60	0.119 65e−42	0.274 08e−50	0.283 42e−53	0.465 11e−57	−0.924 43e−60	0.201 04e−62	0.221 76e−64	−0.511 29e−68	0.299 68e−68
80	−0.220 37e−56	0.480 00e−67	0.367 58e−70	−0.830 46e−76	0.387 59e−79	0.131 57e−82	−0.125 580e−85	0.343 529e−88	0.173 835e−91
100	0.428 79e−70	−0.412 35e−83	0.196 45e−87	0.111 75e−94	−0.999 57e−99	0.645 31e−103	0.772 76e−107	0.827 19e−110	−0.900 45e−113

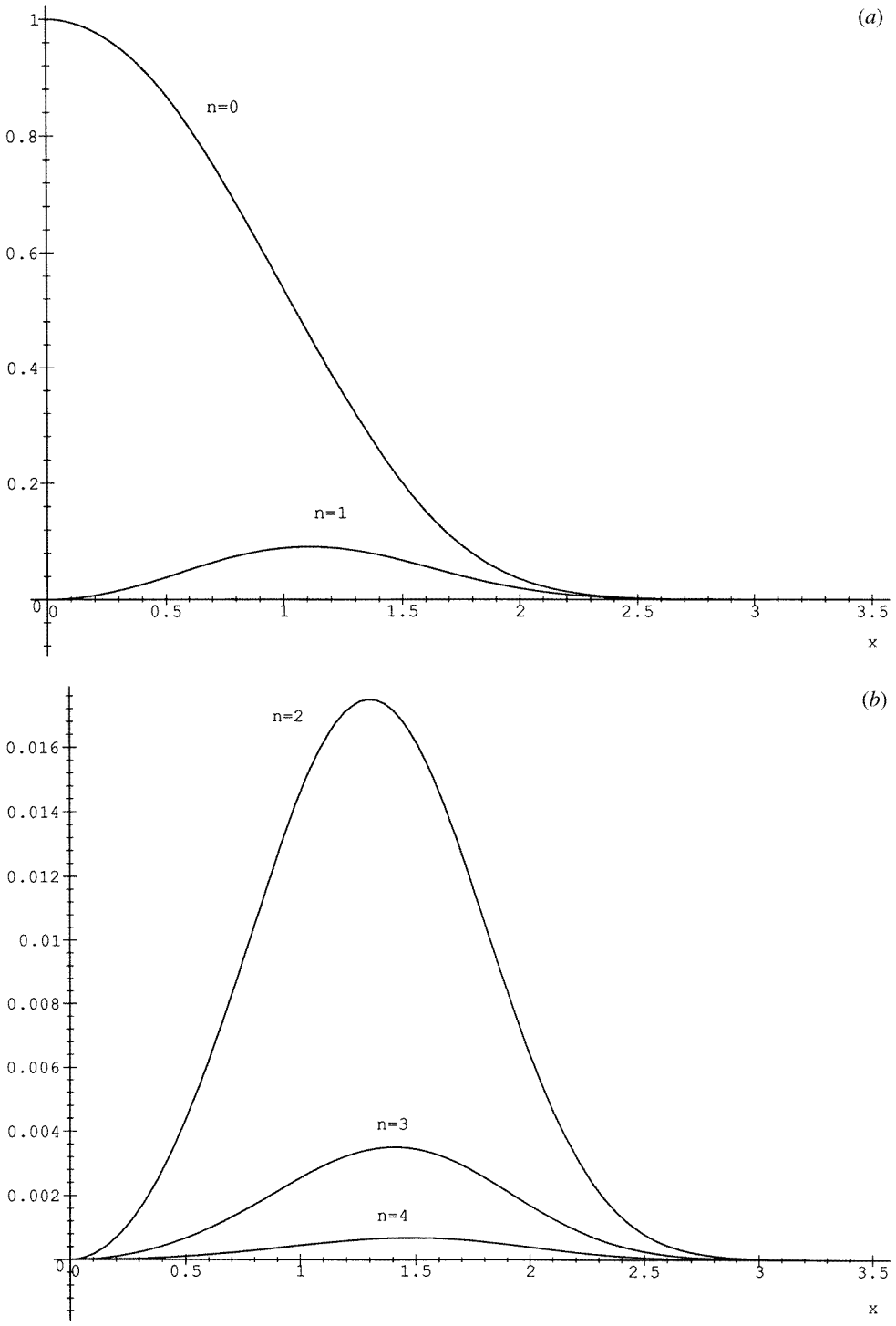

Figure 2. The perturbation wavefunctions ψ_n multiplied by $(-1)^n$ for the ground state of the quartic oscillator with the ordinary Hamiltonian $H = \beta^{1/3}[H_0 + \beta^{-2/3}H_1]$, where $H_0 = p^2 + x^4$ and $H_1 = x^2$. Because of the symmetry of the functions ψ_n, only the $x > 0$ part of the functions is shown. (*a*) $n = 0, 1$, (*b*) $n = 2, \ldots, 4$, (*c*) $n = 5, \ldots, 7$, (*d*) $n = 21, \ldots, 23$. In case (*d*), the functions are multiplied by 10^{10}.

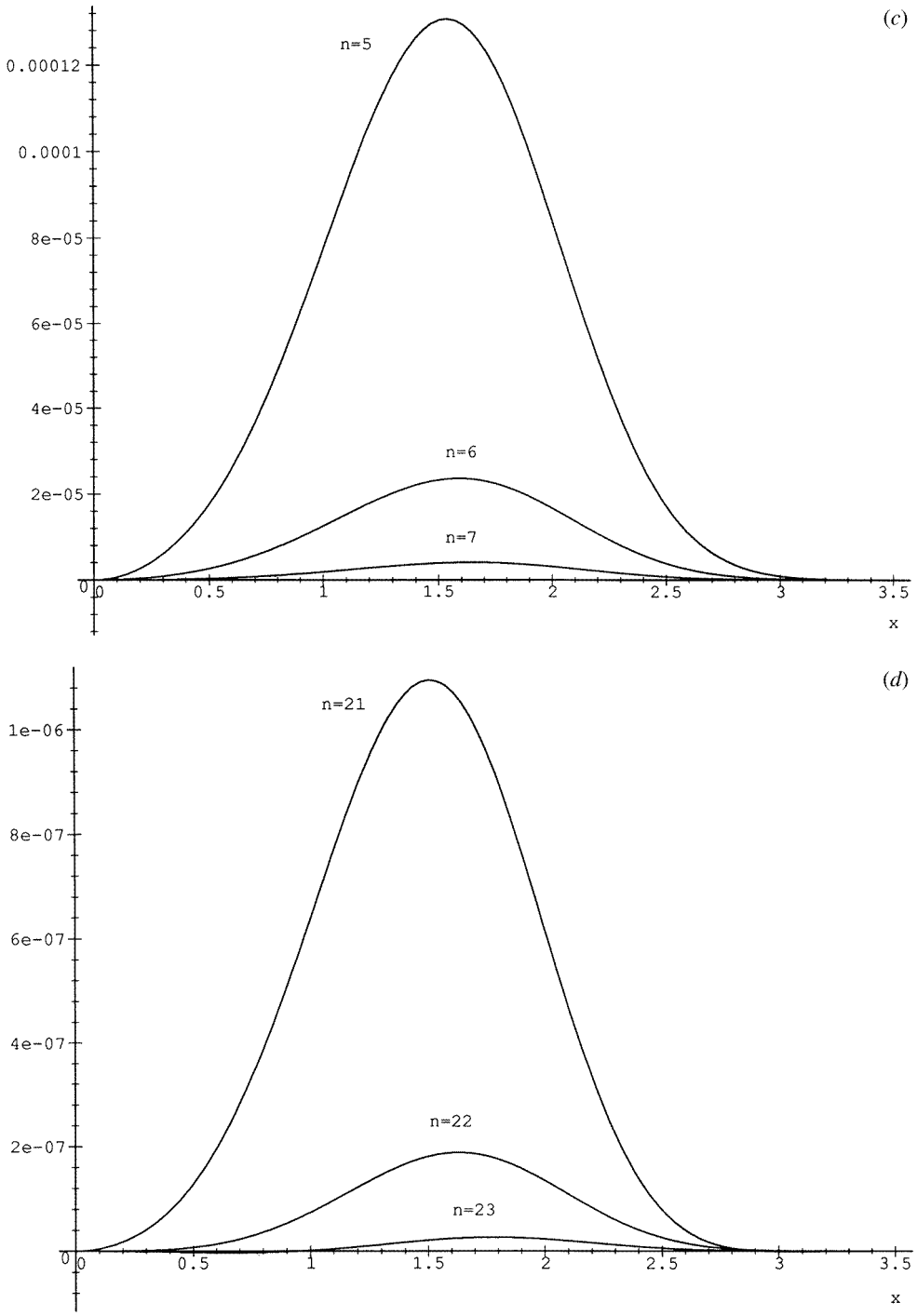

Figure 2. (Continued)

- Truncation of the series (12) yields the upper bounds to the exact value of the energy $E_R(\kappa)$.
- The sum of the truncated series plus the remainder of the series in which equation (14) is used gives the lower bounds to the energy $E_R(\kappa)$.
- The perturbation wavefunctions ψ_n have a simple form which clarifies the sign pattern of the Γ_n coefficients. Truncating the perturbation series (42) one can make the error caused by the truncation arbitrarily small.
- The transformation (6) and the large-order formula (14) depend on the constant B_m. The value (7) of this constant which was optimized for the ground state leads to slower convergence of the series (12) for higher excited states. For these states, another value of the constant B_m can be taken to improve the convergence of the series (12).

In comparison with the renormalized strong coupling expansion (12), the ordinary strong coupling expansion (4) has the following properties:

- The expansion converges for all $\beta \in (\beta_{min}, \infty)$ corresponding to $z \in (0, |z_K|)$ in equation (19). This series converges also for $z \in (-|z_K|, 0)$ corresponding to the double-well potential $V(x) = x^{2m} - |z|x^2$. The value of β_{min} decreases with increasing m and K.
- The large-order formula (21) depends on the branch point z_K and coefficients c_i which are not known analytically.
- The application of this formula to higher excited states is possible for very large n only; for smaller n, the extension of equation (21) to larger number of the branch points is necessary.
- Truncation of the series (4) does not give the upper or lower bounds to the energy $E(\beta)$.
- The perturbation wavefunctions have a more complex form than in the renormalized case and have a less straightforward interpretation.
- With increasing n, the perturbation coefficients K_n go to zero more rapidly than the coefficients Γ_n. However, it does not necessarily mean more rapid convergence of the series (4) with respect to the series (12) since it also depends on the values of the expansion variables $\beta^{-2/(m+1)}$ and $1 - \kappa$.

Our results show that the renormalized strong coupling expansion (12) is the most advantageous perturbative approach to the anharmonic oscillators. At the same time, it represents a non-trivial example of the perturbation theory which converges for all values of the coupling constant $\beta > 0$ and has some other useful properties. Since the anharmonic oscillators are important model systems not only in quantum mechanics and quantum field theory but also in many applications (see e.g. [39–42]), we believe that our results contribute to a deeper understanding of the large-order perturbation theories in general.

Acknowledgments

This work was supported in part by the NSERC and CFCSU of Canada and the GA CR (grant No 202/97/1016) and the GA UK (grant No 155/96) of Czech Republic.

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